Introduction to the Soft - Collinear Effective Theory

Lecture 2

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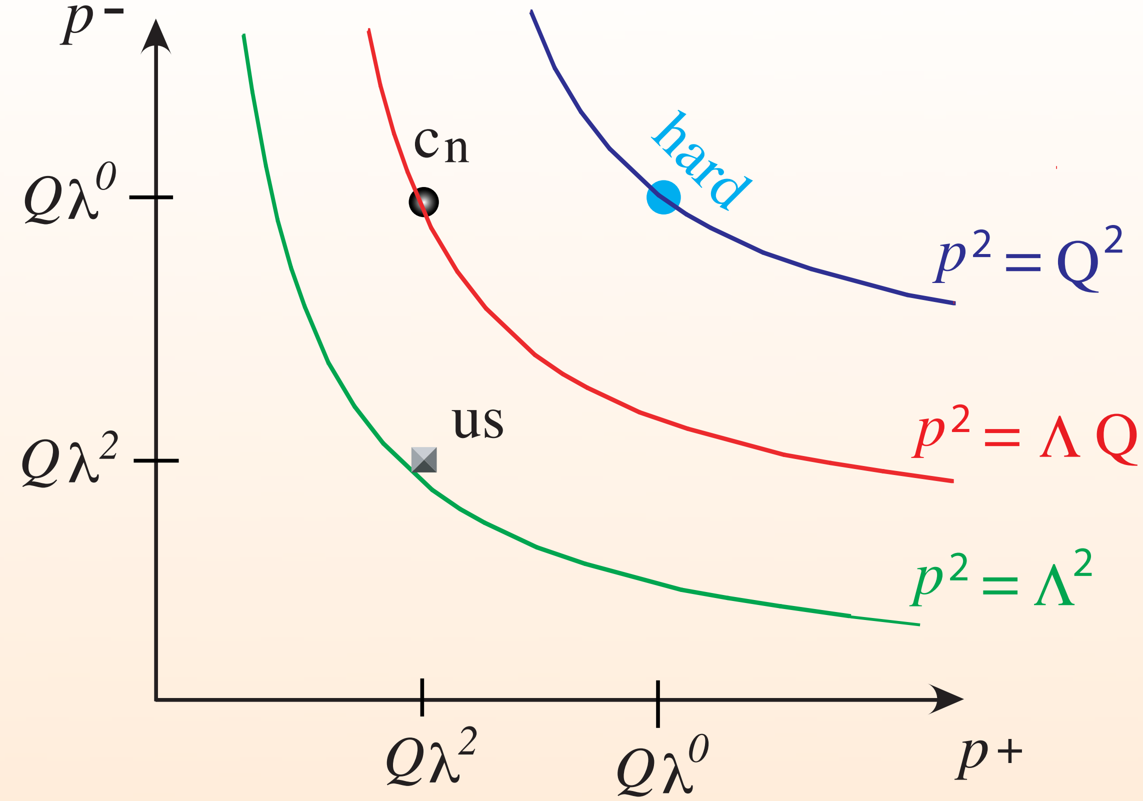
Lets first recall a few things from Lecture 1

SCET$_{\rm I}$ for energetic jets



usoft & collinear modes

$q_{us} \sim \lambda^3$ $\quad\quad \xi_n \sim \lambda$

$A^\mu_{us} \sim \lambda^2$ $\quad\quad (A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, 1, \lambda) \sim p_c^\mu$

two types of derivatives:

$$\mathcal{P}^\mu \xi_{n,p}(x)\,, \quad i\partial^\mu \xi_{n,p}(x)\,, \quad \mathcal{P}^\mu q_{us}(x) = 0\,, \quad i\partial^\mu q_{us}(x)$$

$(1, \lambda)$ $\quad\quad \lambda^2$

$$iD_\perp^{n\mu} = \mathcal{P}_\perp^\mu + gA_n^{\perp\mu}$$

$$i\bar{n}\cdot D_n = \bar{n}\cdot\mathcal{P} + g\bar{n}\cdot A_n$$

$$iD_{us}^\mu = i\partial^\mu + gA_{us}^\mu$$

LO SCET$_{\rm I}$ Lagrangians

$$\mathcal{L}_c^{(0)} = \bar{\xi}_n \Big\{ n\cdot iD_{us} + gn\cdot A_n + i\not{D}_\perp^n \frac{1}{i\bar{n}\cdot D_n} i\not{D}_\perp^n \Big\} \frac{\bar{\not{n}}}{2} \xi_n$$

$$\mathcal{L}_{cg}^{(0)} = \mathcal{L}_{cg}^{(0)}(A_n^\mu, n\cdot A_{us})\,, \quad \mathcal{L}_{us}^{(0)} = \mathcal{L}^{\rm QCD}(q_{us}, A_{us}^\mu)$$

Outline for Lecture 2

- Wilson lines and the heavy-light current
- Gauge Invariance, Reparmaterization Invariance
- Hard-Collinear and Ultrasoft-Collinear Factorization

- SCET Loops, IR divergences, zero-bin
- RGE and Sudakov double logarithms

- $B \to X_s \gamma$ factorization theorem

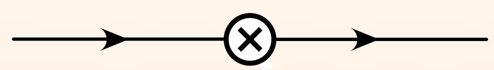
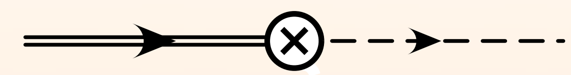
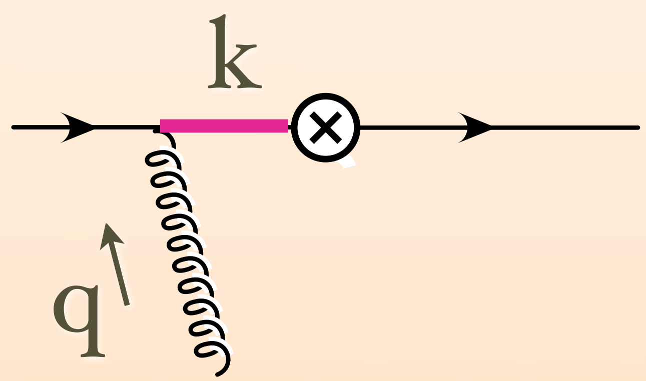
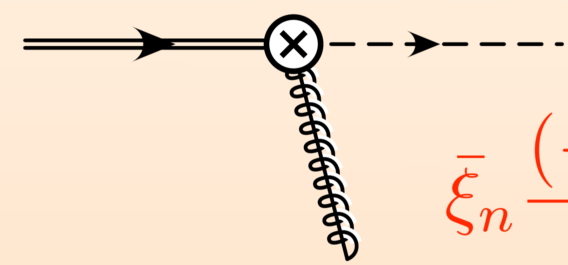
$$(A_n^+, A_n^-, A_n^\perp) \sim (\lambda^2, 1, \lambda) \sim p^\mu$$

We can build LO operators with any number of A_n^- fields.

Should we be concerned?

Currents

eg. $\bar{u}\,\Gamma\, b$ involves both collinear and usoft objects

	QCD		SCET
no gluons	 $\bar{u}\,\Gamma\, b$	→	 $\bar{\xi}_n \;\Gamma\; h_v$
one gluon		→	 $\bar{\xi}_n \dfrac{(-g\bar{n}\cdot A_n)}{\bar{n}\cdot q}\Gamma h_v$

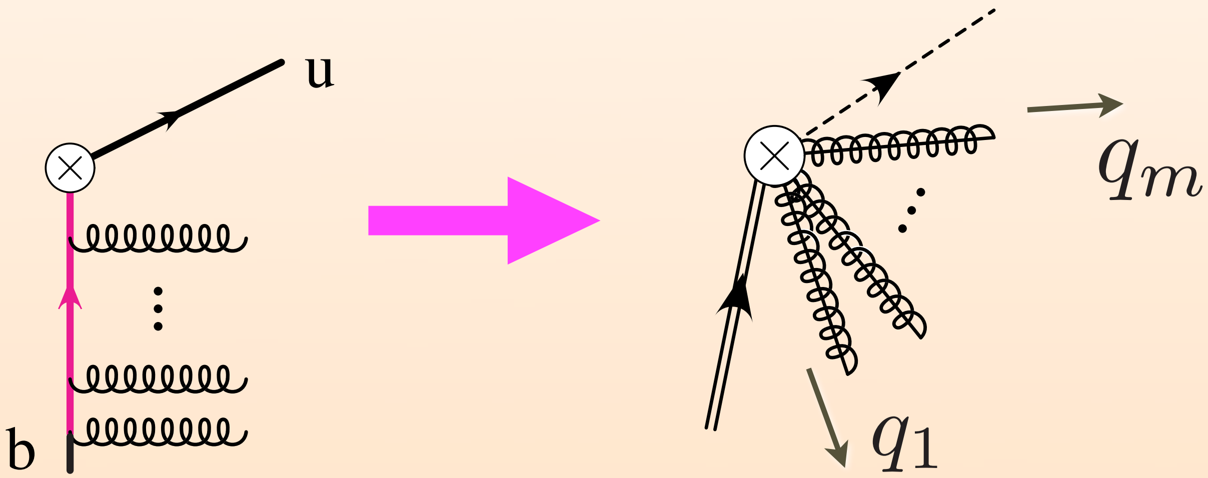
offshell
$$k^\mu = m_b v^\mu + \frac{n^\mu}{2}\bar{n}\cdot q + \ldots$$
$$k^2 - m_b^2 = n\cdot v\, m_b \bar{n}\cdot q + \ldots \;\sim \lambda^0$$

graph =
$$\bar{u}_n \Gamma \frac{i(\not{k} + m_b)}{k^2 - m_b^2} i g T^A \gamma^\mu u_v = \frac{-g}{n\cdot v\, m_b\, \bar{n}\cdot q}\,\bar{u}_n\Gamma\Big[m_b(1+\not{v}) + \frac{\not{n}}{2}\bar{n}\cdot q\Big]\Big(\frac{\not{\bar{n}}}{2}\bar{n}^\mu\Big)T^A u_v$$
$$= \frac{-g\bar{n}^\mu}{\bar{n}\cdot q}\,\bar{u}_n\Gamma T^A\left\{\frac{\frac{\not{\bar{n}}}{2}(1-\not{v}) + n\cdot v + 0}{n\cdot v}\right\}u_v \quad = \frac{-g\bar{n}^\mu}{\bar{n}\cdot q}\,\bar{u}_n\Gamma T^A u_v$$

Currents eg. $\bar{u}\,\Gamma\,b$ involves both collinear and usoft objects

now add any number of gluons

$\bar{u}\,\Gamma\,b \;\longrightarrow\; \bar{\xi}_n W\;\Gamma\;h_v$ get a Wilson line



$$= g^m \sum_{\text{perms}} \frac{(\bar{n}^{\mu_m} T^{A_m})\cdots(\bar{n}^{\mu_1} T^{A_1})}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^m q_i]}$$

$\sim \lambda^0$ no cost to add these gluons

momentum space Wilson line
$$W = \sum_k \sum_{\text{perms}} \frac{(-g)^k}{k!} \left(\frac{\bar{n}\cdot A_{\bar{n},q_1} \cdots \bar{n}\cdot A_{\bar{n},q_k}}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^k q_i]} \right)$$

position space Wilson line
$$W(y,-\infty) = P\exp\Big(ig\int_{-\infty}^{y} ds\,\bar{n}\cdot A_n(s\bar{n}^\mu)\Big)$$

Exercise

SCET Operators with Collinear Quarks and Wilson Lines

a) Start with the QCD Lagrangian for a massive quark and decompose $\not{D}$ in terms of n, $\bar{n}$, and $\perp$ components. As in lecture, write $\psi = \xi_n + \zeta_{\bar{n}}$ where $\not{n}\xi_n = 0$ and $\not{\bar{n}}\zeta_{\bar{n}} = 0$ and determine which products of fields are non-zero. Keeping all the non-zero terms, integrate out the field $\zeta_{\bar{n}}$ to generate an effective action for the massive collinear quark ξ_n.
[With power counting $m \sim p_\perp \sim Q\lambda \ll Q$ this is the starting point to derive the action for a massive collinear quark, ie. prior to decomposing the gluon field into collinear and ultrasoft pieces and prior to distinguishing between large and small momenta. The remaining steps are the same as those discussed in lecture except that you keep the mass. The mass terms that you have derived are important for considering how a collinear Lagrangian of light quarks u, d, s explicitly breaks chiral symmetry. They are also relevant for discussing an energetic jet initiated by a massive quark, when the jet energy $Q \gg m$.]

b) To get more familiar with Wilson lines lets consider the current for a $b \to u$ transition. In QCD $J = \bar{u}\Gamma b$. For SCET we did a matching calculation to find the leading order current

$$J^{(0)} = \bar{\xi}_n W \Gamma h_v \,, \tag{2}$$

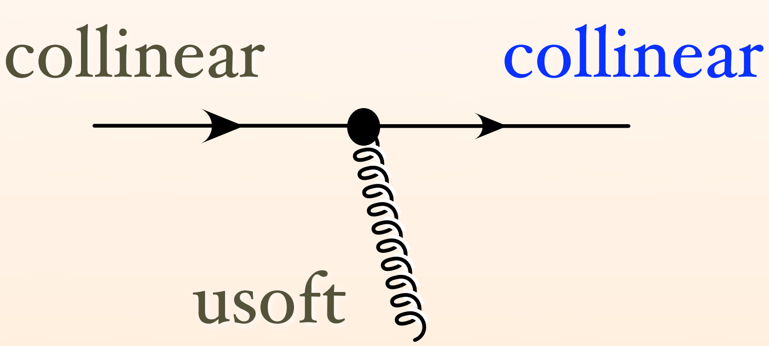
where W included terms involving the order λ^0 collinear gluon field $\bar{n}\cdot A_n$. In lecture we explicitly computed the term in W with one $\bar{n}\cdot A_n$ field and wrote down the result for any number of $\bar{n}\cdot A_n$ fields. Do the matching computation for two $\bar{n}\cdot A_n$ fields (by expanding QCD diagrams with offshell propagators). Verify that the result for one and two $\bar{n}\cdot A_n$ fields agree with the momentum space Feynman rules derived from the position space Wilson line

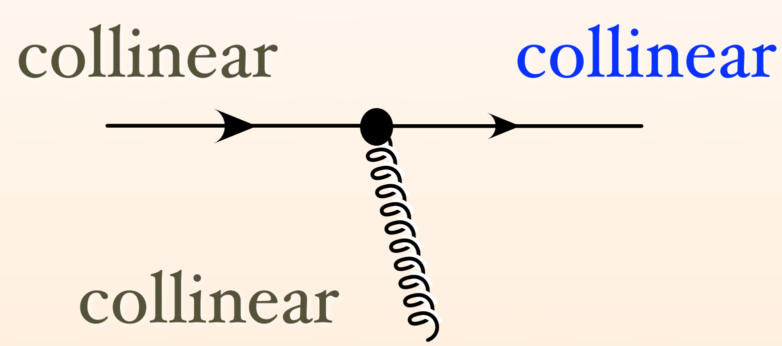
$$W(y^+) = P \exp\left(ig \int_{-\infty}^{0} ds\, \bar{n}\cdot A_n(s\bar{n} + y^+) \right),$$

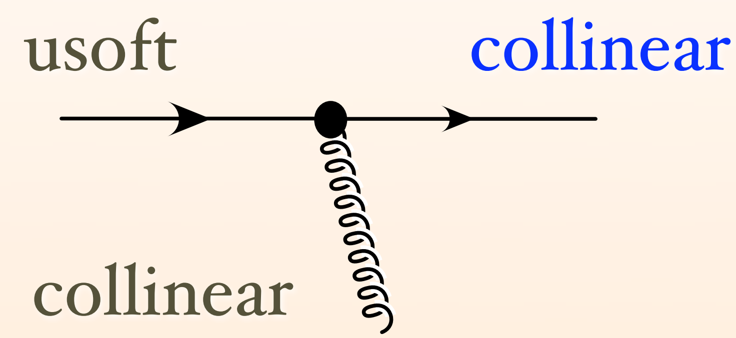
where P is path-ordering.

Interaction of modes: Offshell versus Onshell

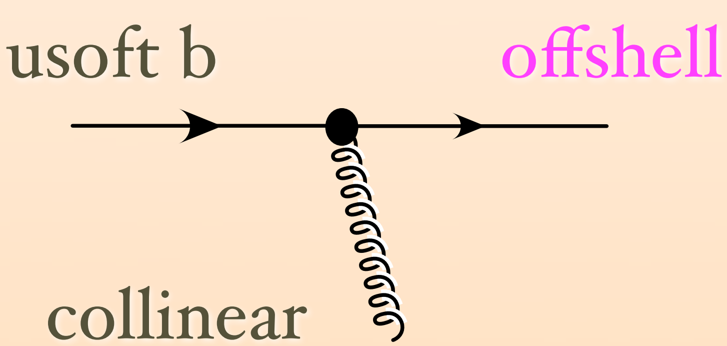
Which fields can interact in a local way?





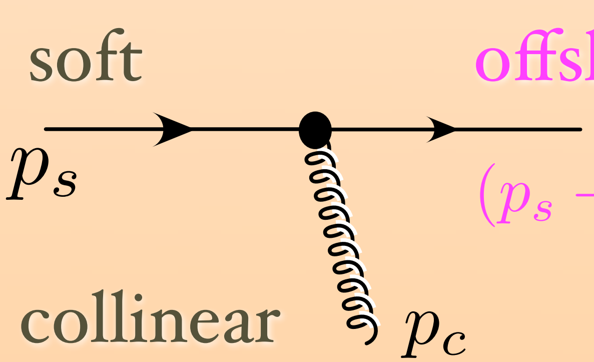


these three are all in $\mathrm{SCET_I}$



this generated the Wilson line W in the $\mathrm{SCET_I}$ computation we just discussed

$\mathrm{SCET_{II}}$: $p_s^2, p_c^2 \sim \lambda^2$



$(p_s + p_c)^2 = p_c^- p_s^+ \sim \lambda$

This makes interactions in $\mathrm{SCET_{II}}$ more complicated to construct, so we postponed further discussion to after fully developing $\mathrm{SCET_I}$

Our analysis of the Lagrangian and Current was tree level.

To determine what effect loops can have we will use Symmetries:

Gauge symmetry

Lorentz invariance (?)

(plus of course Power Counting)

Gauge symmetry

$U(x) = \exp\left[i\alpha^A(x) T^A\right]$

need to consider U's which leave us in the EFT

collinear $\quad i\partial^\mu \mathcal{U}_c(x) \sim p_c^\mu \mathcal{U}_c(x) \leftrightarrow A^\mu_{n,q}$

usoft $\quad i\partial^\mu U_{us}(x) \sim p_{us}^\mu U_{us}(x) \leftrightarrow A^\mu_{us}$

Object	Collinear $\mathcal{U}_c$	Usoft U_{us}
ξ_n	$\mathcal{U}_c\, \xi_n$	$U_{us}\, \xi_n$
gA_n^μ	$\mathcal{U}_c\, gA_n^\mu\, \mathcal{U}_c^\dagger + \mathcal{U}_c\left[i\mathcal{D}^\mu, \mathcal{U}_c^\dagger\right]$	$U_{us}\, gA_n^\mu\, U_{us}^\dagger$
W	$\mathcal{U}_c\, W$	$U_{us}\, W\, U_{us}^\dagger$
q_{us}	q_{us}	$U_{us}\, q_{us}$
gA_{us}^μ	gA_{us}^μ	$U_{us} gA_{us}^\mu U_{us}^\dagger + U_{us}[i\partial^\mu, U_{us}^\dagger]$
Y	Y	$U_{us}\, Y$

Connects:

$$iD_\perp^{n\mu} = \mathcal{P}_\perp^\mu + gA_n^{\perp\mu} \qquad iD_{us}^\mu = i\partial^\mu + gA_{us}^\mu$$

$$i\bar{n}\cdot D_n = \bar{n}\cdot\mathcal{P} + g\bar{n}\cdot A_n$$

$$in\cdot\partial + gn\cdot A_n + gn\cdot A_{us}$$

in the table: $i\mathcal{D}^\mu \equiv \frac{n^\mu}{2}\bar{n}\cdot\mathcal{P} + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2}(in\cdot\partial + gn\cdot A_{us})$

Gauge symmetry

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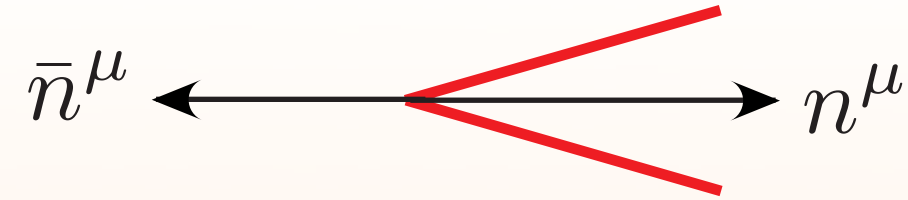
gA^μ_n	$\mathcal{U}_c\, gA^\mu_n\, \mathcal{U}_c^\dagger + \mathcal{U}_c\left[i\mathcal{D}^\mu, \mathcal{U}_c^\dagger\right]$	$U_{us}\, gA^\mu_n\, U_{us}^\dagger$
W	$\mathcal{U}_c\, W$	$U_{us}\, W\, U_{us}^\dagger$
q_{us}	q_{us}	$U_{us}\, q_{us}$
gA^μ_{us}	gA^μ_{us}	$U_{us} g A^\mu_{us} U_{us}^\dagger + U_{us}[i\partial^\mu, U_{us}^\dagger]$
Y	Y	$U_{us}\, Y$

our current is invariant:

$$(\bar{\xi}_n W)\Gamma h_v \longrightarrow (\bar{\xi}_n \mathcal{U}_c^\dagger \mathcal{U}_c W)\Gamma h_v = (\bar{\xi}_n W)\Gamma h_v$$

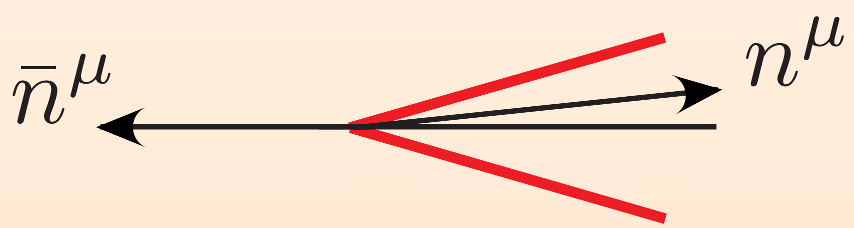
$$\longrightarrow (\bar{\xi}_n U_{us}^\dagger U_{us} W) U_{us}^\dagger \Gamma U_{us} h_v = (\bar{\xi}_n W)\Gamma h_v$$

Reparameterization Invariance (RPI)

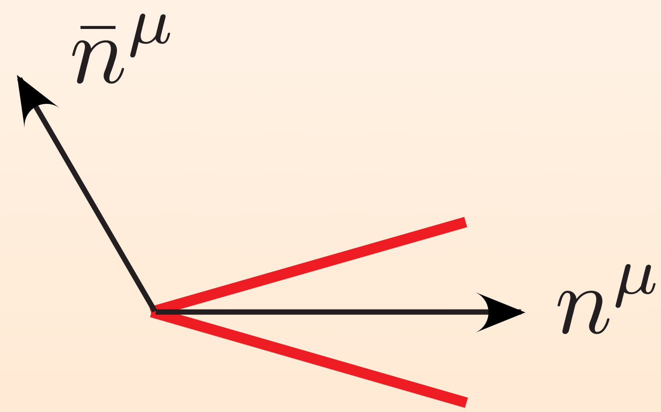


$n\ ,\ \bar{n}$ break Lorentz invariance, restored within collinear cone by reparameterization transformations that preserve power counting. Three types:

(I)



(II)



simultaneous rescaling (longitudinal boost)

$$(\mathrm{I}) \begin{cases} n_\mu \to n_\mu + \Delta_\mu^\perp \\ \bar{n}_\mu \to \bar{n}_\mu \end{cases} \qquad (\mathrm{II}) \begin{cases} n_\mu \to n_\mu \\ \bar{n}_\mu \to \bar{n}_\mu + \varepsilon_\mu^\perp \end{cases} \qquad (\mathrm{III}) \begin{cases} n_\mu \to (1+\alpha)\, n_\mu \\ \bar{n}_\mu \to (1-\alpha)\, \bar{n}_\mu \end{cases}$$

$\Delta_\mu^\perp \sim \lambda$ $\qquad$ $\varepsilon_\mu^\perp \sim \lambda^0$ $\qquad$ $\alpha \sim \lambda^0$

unique

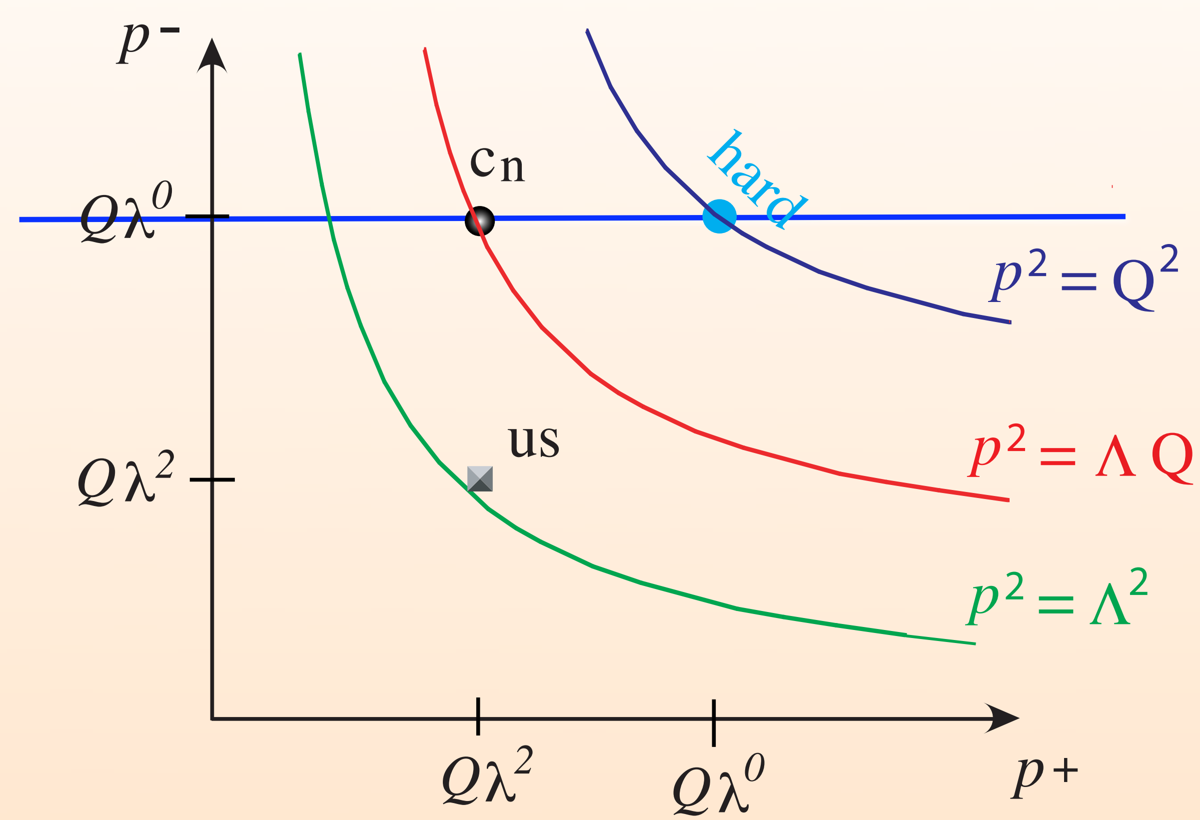
$$\mathcal{L}_c^{(0)} = \bar{\xi}_n \Big\{ n \cdot iD_{us} + g n \cdot A_n + i \not{D}_\perp^c \frac{1}{i\bar{n} \cdot D_c} i \not{D}_\perp^c \Big\} \frac{\not{\bar{n}}}{2} \xi_n$$

Factorization from SCET

Wilson Coefficients and Hard - Collinear Factorization

$$\begin{array}{lll} & & \overline{(+,-,\perp)} \\ \text{hard:} & p^\mu \sim & (1\,,1\,,1) \\ \text{collinear:} & p^\mu \sim & (\lambda^2,1,\lambda) \end{array}$$

can exchange momenta



Constrained by gauge invariance:

$C(\bar{\mathcal{P}},\mu)$: they depend on large momenta picked out by $\bar{\mathcal{P}} = \bar{n}\cdot\mathcal{P} \sim \lambda^0$

eg. $$C(-\bar{\mathcal{P}},\mu)\,(\bar{\xi}_n W)\Gamma h_v = \underbrace{(\bar{\xi}_n W)}\Gamma h_v\, C(\bar{\mathcal{P}}^\dagger,\mu)$$

only the product
is gauge invariant

implies convolutions between coefficients and operators

Write

$$(\bar{\xi}_n W)\Gamma h_v\, C(\bar{\mathcal{P}}^\dagger,\mu) = \int d\omega\, C(\omega,\mu)\Big[\underbrace{(\bar{\xi}_n W)\delta(\omega-\bar{\mathcal{P}}^\dagger)}\Gamma h_v\Big] = \int d\omega\, C(\omega,\mu) O(\omega,\mu)$$

$$\equiv (\bar{\xi}_n W)_\omega\, \Gamma h_v$$

hard-collinear factorization follows from properties of SCET operators

In general:

$$f(i\bar{n}\cdot D_c) = W f(\bar{\mathcal{P}}) W^\dagger$$

$$= \int d\omega\; f(\omega)\; \left[W\delta(\omega-\bar{\mathcal{P}})W^\dagger\right]$$

hard coefficient $p^2 \sim Q^2$

in collinear operator $p^2 \sim Q^2\lambda^2$

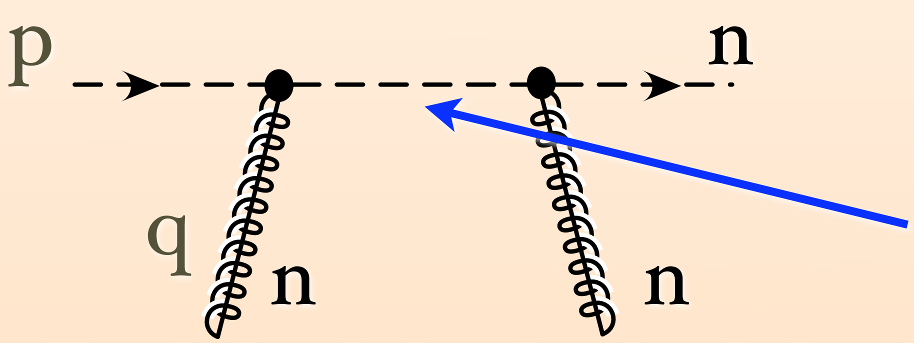
We can trade $\bar{n}\cdot A$ for the Wilson line $W[\bar{n}\cdot A\ ]$

Properties of $\mathcal{L}_c^{(0)} = \bar{\xi}_n \Big\{ n\cdot iD_{us} + g n\cdot A_n + i\not{D}_\perp^n \frac{1}{i\bar{n}\cdot D_n} i\not{D}_\perp^n \Big\} \frac{\not{\bar{n}}}{2} \xi_n$

1) has particles and antiparticles, pair creation & annihilation

$$\frac{i\not{n}}{2}\frac{\theta(\bar{n}\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} + i\epsilon} + \frac{i\not{n}}{2}\frac{\theta(-\bar{n}\cdot p)}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} - i\epsilon} = \frac{i\not{n}}{2}\frac{\bar{n}\cdot p}{n\cdot p\, \bar{n}\cdot p + p_\perp^2 + i\epsilon} = \frac{i\not{n}}{2}\frac{\bar{n}\cdot p}{p^2 + i\epsilon}$$

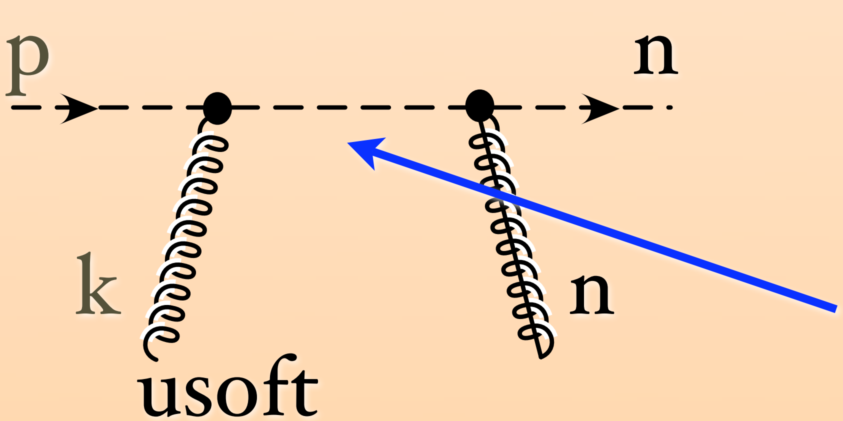
2) all components of A_n^μ couple to ξ_n



$$\frac{i\not{n}}{2}\frac{\bar{n}\cdot(p+q)}{(p+q)^2 + i\epsilon}$$

all components of p & q appear

3) only $n\cdot A_{us}$ couple at LO, only depends on $n\cdot k_{us}$ momentum



$$\frac{i\not{n}}{2}\frac{\bar{n}\cdot p}{\bar{n}\cdot p\, n\cdot(p+k) + p_\perp^2 + i\epsilon} = \frac{i\not{n}}{2}\frac{\bar{n}\cdot p}{\bar{n}\cdot p\, n\cdot k + p^2 + i\epsilon}$$

$$= \frac{i\not{n}}{2}\frac{1}{n\cdot k + i\epsilon}$$

eikonal

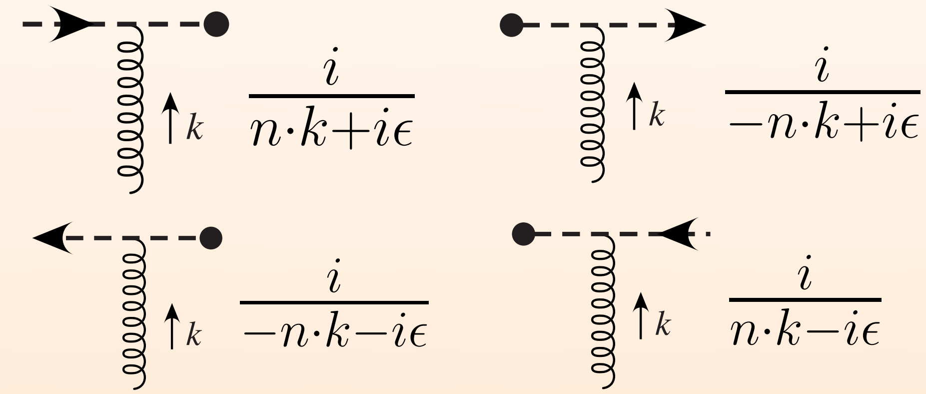
onshell $p^2 = 0$

Ultrasoft - Collinear Factorization

Multipole Expansion:

$$\mathcal{L}_c^{(0)} = \bar{\xi}_n \Big\{ n\cdot iD_{us} + gn\cdot A_n + i\not{D}_\perp^c \frac{1}{i\bar{n}\cdot D_c} i\not{D}_\perp^c \Big\} \frac{\not{\bar{n}}}{2} \xi_n$$

usoft gluons have eikonal Feynman rules and induce eikonal propagators



Field Redefinition:

$$\xi_n \to Y\xi_n \quad , \quad A_n \to Y A_n Y^\dagger \qquad Y(x) = P\exp\Big(ig\int_{-\infty}^{0} ds\, n\cdot A_{us}(x+ns)\Big)$$

$$n\cdot D_{us} Y = 0,\; Y^\dagger Y = 1$$

choice of $\pm\infty$ here is irrelevant if one is careful

gives:

$$\mathcal{L}_c^{(0)} = \bar{\xi}_n \Big\{ n\cdot iD_{\rm us} + \ldots \Big\} \frac{\not{\bar{n}}}{2} \xi_n \to \bar{\xi}_n \Big\{ n\cdot iD_c + i\not{D}_\perp^c \frac{1}{i\bar{n}\cdot D_c} i\not{D}_\perp^c \Big\} \frac{\not{\bar{n}}}{2} \xi_n$$

Moves all usoft gluons to operators, simplifies cancellations

Field Theory gives the same results pre- and post- field redefinition, but the organization is different

Ultrasoft - Collinear Factorization:

eg1. $J = (\bar{\xi}_n W)_\omega \Gamma h_v \to (\bar{\xi}_n Y^\dagger Y W Y^\dagger)_\omega \Gamma h_v = (\bar{\xi}_n W)_\omega \Gamma (Y^\dagger h_v)$

note: not upset by hard-collinear momentum fraction since ultrasoft gluons carry no hard momenta

so usoft-collinear factorization is also simply a property of SCET

eg2. No ultrasoft fields

$$J = (\bar{\xi}_n W)_{\omega_1} \Gamma (W^\dagger \xi_n)_{\omega_2} \to (\bar{\xi}_n W)_{\omega_1} Y^\dagger Y \Gamma (W^\dagger \xi_n)_{\omega_2} = (\bar{\xi}_n W)_{\omega_1} \Gamma (W^\dagger \xi_n)_{\omega_2}$$

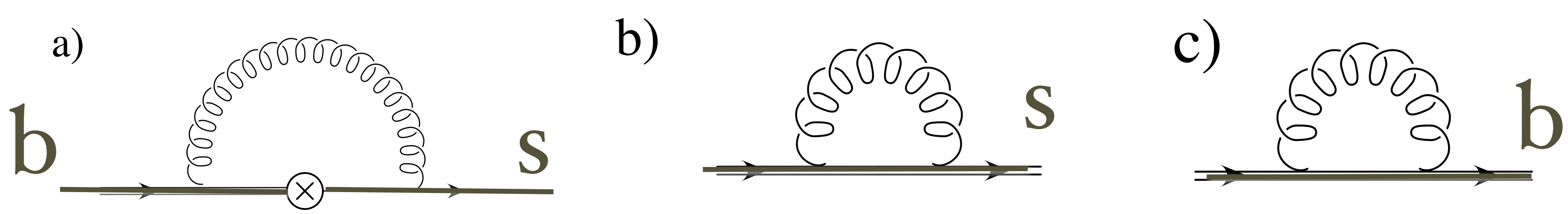
color transparency

Loops in SCET and QCD

Study: IR divergences, UV divergences, and Matching

Consider heavy to light current

QCD $\quad J^{\rm QCD} = \bar{s}\,\Gamma\, b \qquad \Gamma = \sigma^{\mu\nu}$ has UV c.t. $\qquad \bar{n}\cdot p = m_b$



IR regulator

$p^2 \neq 0$ for s-quark

$1/\epsilon_{\rm IR}$ for b-quark

sum= $$-\frac{\alpha_s}{3\pi}\Big[\ln^2\Big(\frac{-p^2}{m_b^2}\Big) + \frac{3}{2}\ln\Big(\frac{-p^2}{m_b^2}\Big) + \frac{1}{\epsilon_{\rm IR}} + 2\ln\Big(\frac{\mu^2}{m_b^2}\Big) + \text{constants}\Big]$$

SCET$_I$

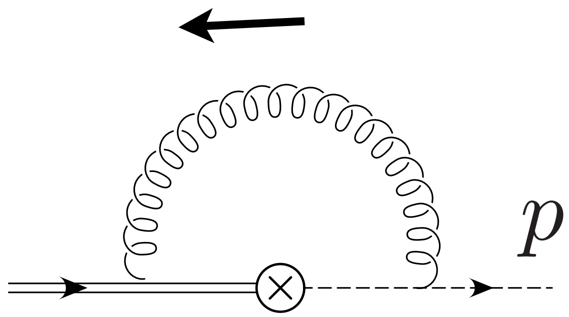
$$J^{\rm SCET} = (\bar\xi_n W)_\omega \Gamma h_v$$

$\bar n\cdot p = m_b$

Feyn. Gauge

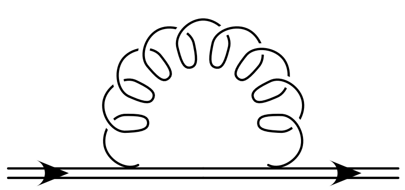
same IR regulators

usoft gluon graphs



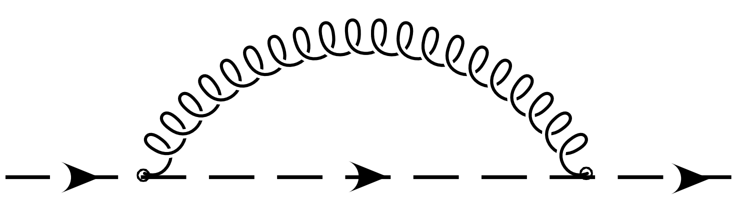
$$C(\bar n\cdot p)\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i0)(v\cdot k+i0)(n\cdot k+p^2/\bar n\cdot p+i0)}$$

$$= -C(\bar n\cdot p)\,(\bar u_n\Gamma u_v)\,\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\ln\left(\frac{\mu\bar n\cdot p}{-p^2}\right)+2\ln^2\left(\frac{\mu\bar n\cdot p}{-p^2}\right)+\ldots\right]$$



$$Z_{h_v} = 1+\frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm IR}}\right)$$

$$C_F = 4/3$$



$$\propto n^2 = 0$$

SCET I

$$J^{\rm SCET} = (\bar\xi_n W)_\omega \Gamma h_v$$

$\bar n\cdot p = m_b$

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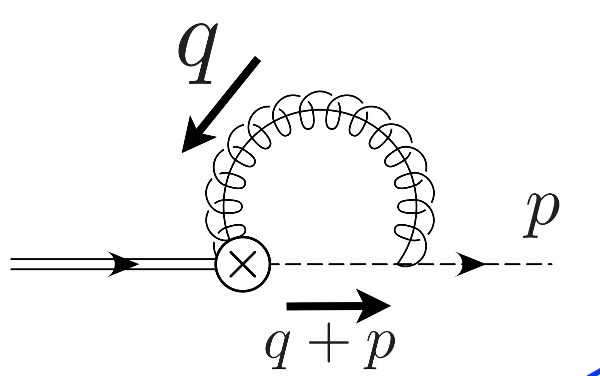
collinear gluon graphs



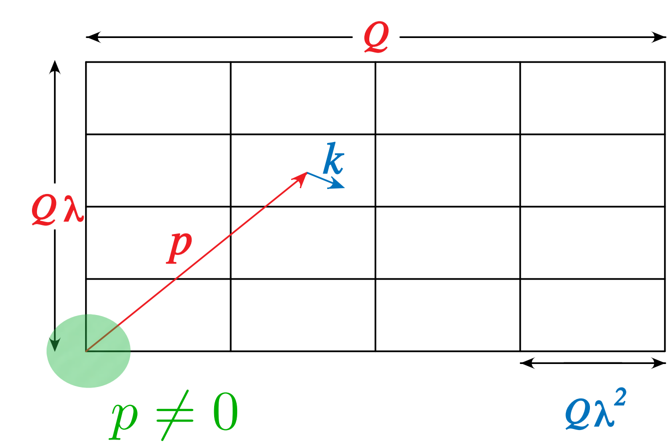
$$\propto C(\bar n\cdot p) \sum_{q\neq 0,\, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \frac{2\bar n\cdot(q+p)}{(\bar n\cdot q + i0)((q+p)^2+i0)(q^2+i0)}$$

$$C\big(\bar n\cdot(p+q) - \bar n\cdot q\big) = C(\bar n\cdot p)$$

$$q = (q, q_r)\,, \quad (q+p)^2 = \bar n\cdot(q+p)\, n\cdot(q_r+p) - (\vec q_\perp + \vec p_\perp)^2$$

"zero-bin" $\displaystyle\sum_{q\neq 0, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \Longrightarrow \int \frac{d^4 q}{(2\pi)^4}$ is okay with our regulator

(more on this later)



so graph $\displaystyle \propto C(\bar n\cdot p) \int \frac{d^4 q}{(2\pi)^4} \frac{2\bar n\cdot(q+p)}{(\bar n\cdot q + i0)((q+p)^2+i0)(q^2+i0)}$

graph $\displaystyle = -C(\bar n\cdot p)\,(\bar u_n \Gamma u_v)\, \frac{\alpha_s C_F}{4\pi} \left[\frac{-2}{\epsilon^2} - \frac{2}{\epsilon} - \frac{2}{\epsilon} \ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right) - 2\ln\left(\frac{\mu^2}{-p^2}\right) + \ldots \right]$

SCET$_{\rm I}$

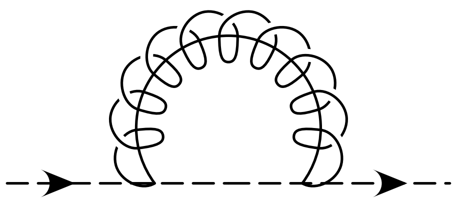
$$J^{\rm SCET} = (\bar{\xi}_n W)_\omega \Gamma h_v$$

$\bar{n}\cdot p = m_b$

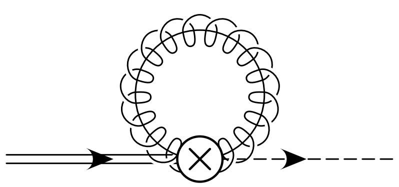
Feyn. Gauge

same IR regulators

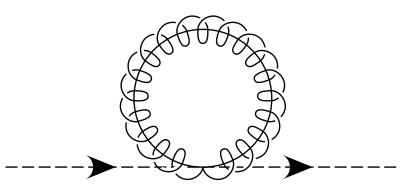
more collinear gluon graphs



$$Z_\xi = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{\rm UV}} + \ln\left(\frac{\mu^2}{-p^2}\right)\right]$$



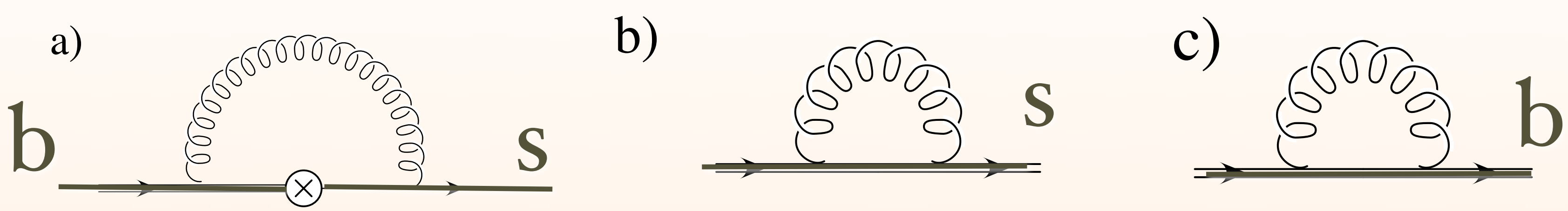
$\propto \bar{n}^2 = 0$



$\propto \bar{n}^2 = 0$

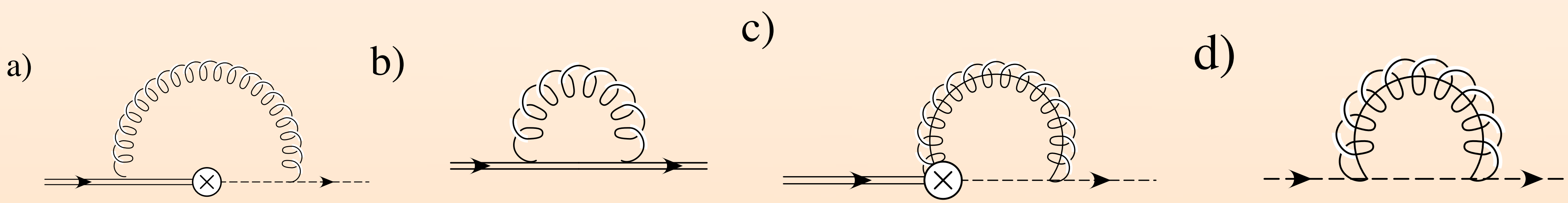
Compare QCD and SCET

QCD



$$\text{sum}= -\frac{\alpha_s}{3\pi}\left[\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{\rm IR}} + 2\ln\left(\frac{\mu^2}{m_b^2}\right) + \text{constants}\right]$$

SCET



same IR divergences

$$\text{sum}= -\frac{\alpha_s}{3\pi}\left[\ln^2\left(\frac{-p^2}{m_b^2}\right) + \frac{3}{2}\ln\left(\frac{-p^2}{m_b^2}\right) + \frac{1}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm UV}^2} - \frac{5}{2\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm UV}}\ln\left(\frac{\mu}{m_b}\right) - 2\ln^2\left(\frac{\mu}{m_b}\right) - \frac{3}{2}\ln\left(\frac{\mu^2}{m_b^2}\right) + \text{constants}\right]$$

UV renormalization in SCET sums double Sudakov logs

remaining terms in SCET & QCD give one-loop matching for $C(\bar{n}\cdot p = m_b, \mu)$

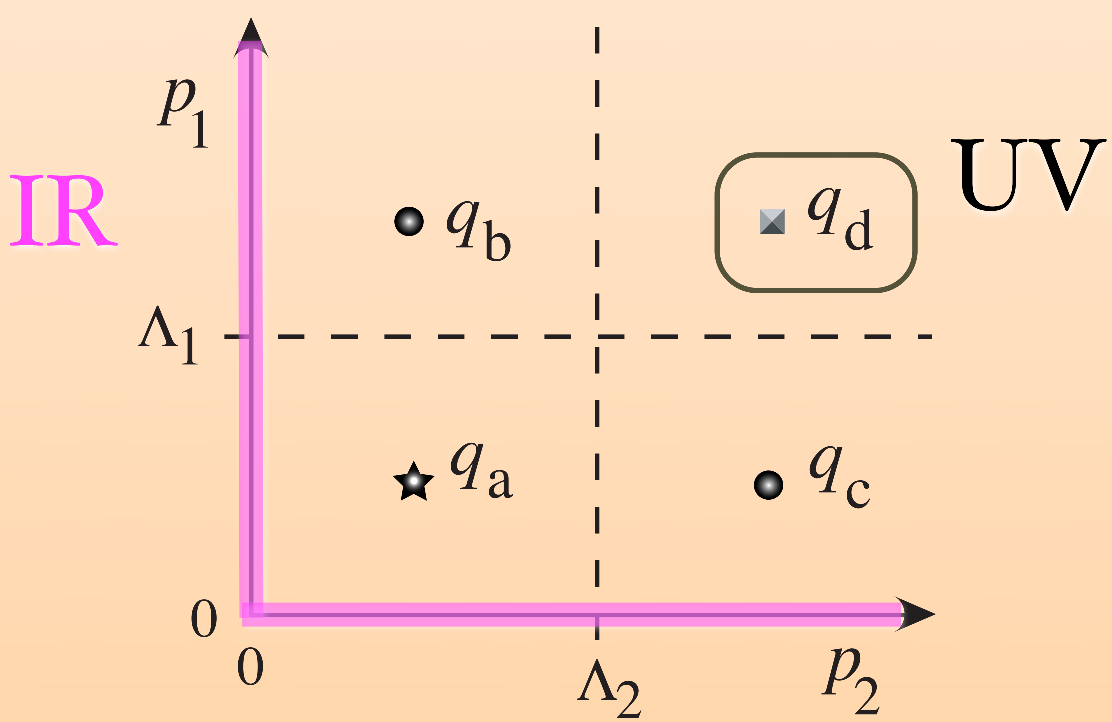
$$\sum_{q\neq 0, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \stackrel{?}{\Longrightarrow} \int \frac{d^4 q}{(2\pi)^4}$$

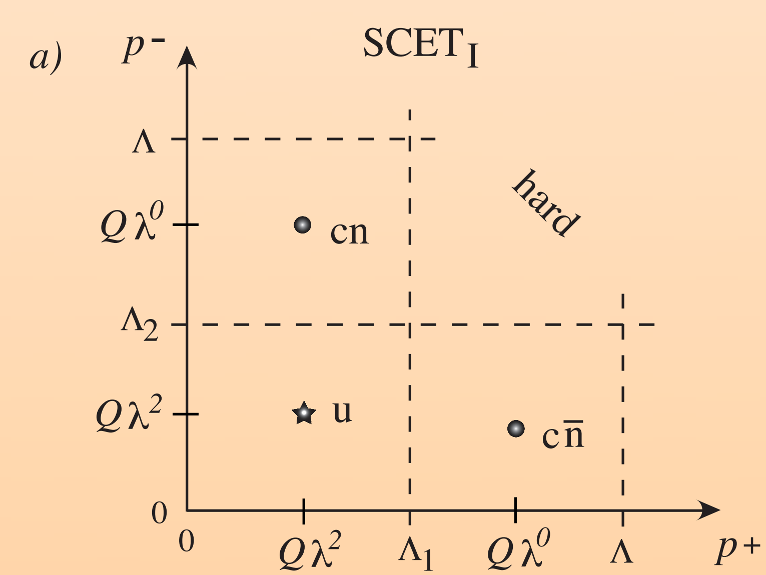
These restrictions ensure that the collinear graph does not double count the IR momentum region taken care of by the usoft graph

General (regulator independent) formula is:

$$\sum_{p_1\neq 0} \int dp_{1r}\, F^{(q_b)}(p_1) = \int dp_1 \Big[F^{(q_b)}(p_1) - F^{(q_b\to q_a)}_{\rm subt}(p_1) \Big]$$

zero-bin subtraction term





For our example

using dim.reg. in UV

$p^2 \neq 0$ in IR

$$\sum_{q\neq 0,\, q\neq -p} \int \frac{d^4 q_r}{(2\pi)^4} \frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)((q+p)^2+i0^+)(q^2+i0^+)}$$

avoids double counting the usoft region

$$= \int \frac{d^d q}{(2\pi)^d} \left[\frac{2\bar{n}\cdot(q+p)}{(\bar{n}\cdot q+i0^+)[(q+p)^2+i0^+](q^2+i0^+)} - \frac{2\bar{n}\cdot p}{(\bar{n}\cdot q+i0^+)[n\cdot q\,\bar{n}\cdot p+p^2+i0^+](q^2+i0^+)} \right]$$

subtraction

$$= -\frac{i}{16\pi^2}\left[-\frac{2}{\epsilon_{\rm IR}\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}} \ln\left(\frac{\mu^2}{-p^2}\right) - \ln^2\left(\frac{\mu^2}{-p^2}\right) + \left(\frac{2}{\epsilon_{\rm IR}} - \frac{2}{\epsilon_{\rm UV}}\right)\ln\left(\frac{\mu}{\bar{n}\cdot p}\right) + \ldots \right.$$
$$\left. - \left(\frac{2}{\epsilon_{\rm UV}} - \frac{2}{\epsilon_{\rm IR}}\right)\left\{\frac{1}{\epsilon_{\rm UV}} + \ln\left(\frac{\mu^2}{-p^2}\right) - \ln\left(\frac{\mu}{\bar{n}\cdot p}\right)\right\} \right]$$

- singularity from $\bar{n}\cdot q \to 0$ cancels between the two terms
- UV collinear singularity comes from $\bar{n}\cdot q \to \infty$ (in subtraction term)
- standard calc. tool of taking $\epsilon_{\rm IR} = \epsilon_{\rm UV}$ with no subtraction gives the same answer

eg. of another regulator

Cutoffs: $\Omega_\perp^2 \leq \vec{q}_\perp^{\,2} \leq \Lambda_\perp^2$ $\quad \Omega_-^2 \leq (q^-)^2 \leq \Lambda_-^2$

no constraint on q^+ , p onshell

QCD

$$I_{\rm full}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big)\Big] + \ldots$$

SCET

$$I_{\rm us}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\ln\Big(\frac{\Omega_-\Lambda_-}{\Omega_\perp^2}\Big)\Big]$$

$$I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{p^-}\Big)\Big] - \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Omega_-}{\Lambda_-}\Big)\Big] = \frac{i}{8\pi^2}\Big[-\ln\Big(\frac{\Omega_\perp^2}{\Lambda_\perp^2}\Big)\ln\Big(\frac{\Lambda_-}{p^-}\Big)\Big] + \ldots$$

$$I_{\rm us}^{b\to s\gamma} + I_{\rm C}^{b\to s\gamma} = \frac{i}{8\pi^2}\Big[\mathrm{Li}_2\Big(\frac{-\Omega_\perp^2}{\Omega_-^2}\Big) + \ln\Big(\frac{\Omega_-}{p^-}\Big)\ln\Big(\frac{\Omega_- p^-}{\Omega_\perp^2}\Big) + \ln^2\Big(\frac{\Lambda_\perp}{p^-}\Big) - \ln^2\Big(\frac{\Lambda_\perp}{\Lambda_-}\Big)\Big] + \ldots$$

$$p^- = m_b$$

IR matches again

but ONLY with the non-zero subtraction term included

Renormalization in SCET
&
Summing Sudakov Logs

Renormalize Heavy to Light Current in SCET

$$C(\omega,\mu)\big[(\bar\xi_n W)_\omega\,\Gamma h_v\big] \qquad C^{\rm bare} = C + (Z_c-1)C \qquad \omega = m_b$$

graph sum = $-\dfrac{\alpha_s}{3\pi}\Big[\ln^2\Big(\dfrac{-p^2}{m_b^2}\Big)+\dfrac{3}{2}\ln\Big(\dfrac{-p^2}{m_b^2}\Big)+\dfrac{1}{\epsilon_{\rm IR}}$

$$-\frac{1}{\epsilon_{\rm UV}^2}-\frac{5}{2\epsilon_{\rm UV}}-\frac{2}{\epsilon_{\rm UV}}\ln\Big(\frac{\mu}{m_b}\Big)-2\ln^2\Big(\frac{\mu}{m_b}\Big)-\frac{3}{2}\ln\Big(\frac{\mu^2}{m_b^2}\Big)+\text{constants}\Big]$$

need $\quad Z_c = 1-\dfrac{\alpha_s(\mu)C_F}{4\pi}\Big(\dfrac{1}{\epsilon^2}+\dfrac{5}{2\epsilon}+\dfrac{2}{\epsilon}\ln\dfrac{\mu}{\omega}\Big)$ to remove UV divergences

Compute the Anomalous Dimension

$$\mu\frac{d}{d\mu}C^{\rm bare}=0 \implies \underline{\mu\frac{d}{d\mu}C(\omega,\mu)=\gamma_c(\omega,\mu)C(\omega,\mu)}$$

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = -2\epsilon\alpha_s(\mu)+\beta[\alpha_s]$$

$$\gamma_c = -Z_c^{-1}\mu\frac{d}{d\mu}Z_c = \mu\frac{d}{d\mu}\frac{\alpha_s(\mu)C_F}{4\pi}\Big(\frac{1}{\epsilon^2}+\frac{5}{2\epsilon}+\frac{2}{\epsilon}\ln\frac{\mu}{\omega}\Big)$$

$$=\frac{\alpha_s(\mu)C_F}{4\pi}\Big(\frac{-2}{\epsilon}-5-4\ln\frac{\mu}{\omega}+\frac{2}{\epsilon}\Big) = -\frac{\alpha_s(\mu)C_F}{\pi}\Big(\ln\frac{\mu}{\omega}+\frac{5}{4}\Big)$$

(the $\frac{-2}{\epsilon}$ and $\frac{2}{\epsilon}$ terms are struck through)

LL (arrow to $\ln\frac{\mu}{\omega}$) — part of NLL (arrow to $\frac{5}{4}$)

LL solution

cusp anomalous dimension

Solve $$\mu \frac{d}{d\mu} \ln C(\omega, \mu) = -\frac{\alpha_s(\mu) C_F}{\pi} \ln \frac{\mu}{\omega} \,, \qquad \mu \frac{d}{d\mu} \alpha_s \;\; \mu \;\; \square - \frac{\beta_0}{\pi} \alpha_s^2 \;\; \mu$$

use $d \ln(\mu) = -\frac{2\pi}{\beta_0} \frac{d\alpha_s}{\alpha_s^2}$ and integrate to obtain the solution

$$C(\omega, \mu) = C(\omega, \mu_0) \exp\left[\frac{-4\pi C_F}{\beta_0^2 \alpha_s(\mu_0)} \left(\frac{1}{z} - 1 + \ln z \right) \right] \left(\frac{\mu_0}{\omega} \right)^{2 C_F \ln z / \beta_0}$$

boundary condition, no large logs for $\mu_0 \sim \omega$

$\sim \exp(\alpha_s \ln^2 + \alpha_s^2 \ln^3 + \ldots)$

$$z \equiv \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

If $\beta_0 \to 0$ and $\alpha_s = \text{constant}$, then

$$C(\omega, \mu) = C(\omega, \mu_0) \exp\left[\frac{-\alpha_s C_F}{\pi} \left(\frac{1}{2} \ln^2 \frac{\mu}{\mu_0} + \ln \frac{\mu}{\mu_0} \ln \frac{\mu_0}{\omega} \right) \right]$$

Sudakov double logs exponentiated

Exercise

SCET Loops for Two-Jet Production

Consider the two-jet production process through a virtual photon in SCET, namely $e^+e^- \to J_n J_{\bar{n}} X_{us}$ where J_n is a jet in the $n = (1,0,0,-1)$ direction, $J_{\bar{n}}$ is a jet in the $\bar{n} = (1,0,0,1)$ direction, and any remaining particles in the final state are ultrasoft, contained in X_{us}.

a) Write down two collinear quark Lagrangians, one for ξ_n fields and one for $\xi_{\bar{n}}$ fields. Interactions between these two types of collinear fields are hard, and so do not effect your analysis. What are the Feynman rules for the ultrasoft gluon coupling to each of these collinear quarks?

b) Start with $J^{\rm QCD} = \bar{\psi}\gamma_\mu\psi$ and determine the appropriate LO SCET current $J^{\rm SCET} = \bar{\xi}_n \cdots \xi_{\bar{n}}$, ie. fill in the dots with appropriate collinear Wilson lines and Dirac structure.

c) Draw the five one-loop Feynman diagrams that are non-zero for $e^+e^- \to q_n \bar{q}_{\bar{n}}$ (use Feynman gauge for all gluons when determining which graphs are zero). Here q_n has n-collinear momentum p, and $\bar{q}_{\bar{n}}$ has $\bar{n}$-collinear momentum $\bar{p}$ and you should work in the CM frame. All graphs but one can be directly read off using the loop computations done in lecture (or given in the handout notes), as long as you use the same IR regulator. That is, you should keep both collinear quarks offshell, $p^2 \neq 0$ and $\bar{p}^2 \neq 0$. Compute the divergent terms in the one remaining ultrasoft graph using dimensional regularization in the UV.

d) Add up the $1/\epsilon$ terms from the graphs in c) and determine the lowest order anomalous dimension equation for C the Wilson coefficient of $J^{\rm SCET}$. Solve this equation keeping only the $\ln\mu/Q$ term and using a fixed coupling α_s, and then with a running coupling $\alpha_s(\mu)$. (Voilá, Sudakov double logs resummed.)

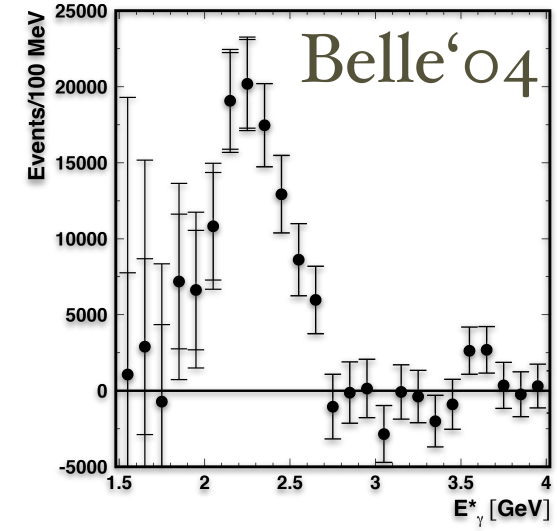
Lets use our LO heavy-to-light current

$$J^{(0)} = \int d\omega\, C(\omega,\mu) \Big[(\bar{\xi}_n W)\delta(\omega - \bar{\mathcal{P}}^\dagger)\Gamma(Y_n^\dagger h_v) \Big] = \int d\omega\, C(\omega,\mu)\ \bar{\chi}_{n,\omega}\, \Gamma\, \mathcal{H}_v^n$$

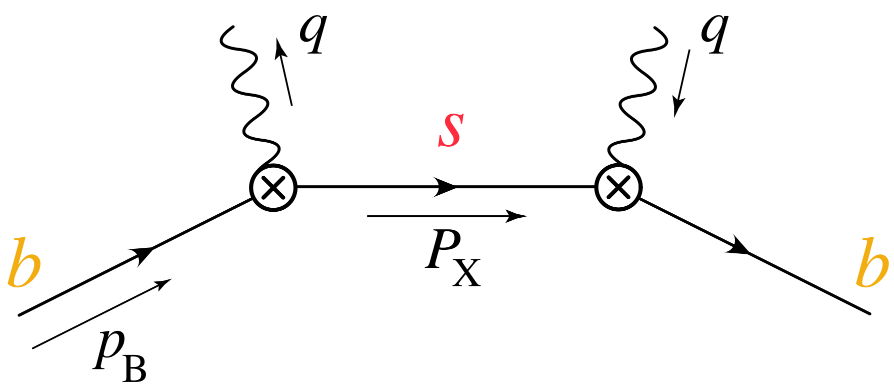
to derive a factorization theorem for the jet-like region of

$$B \to X_s \gamma$$

Endpoint $B \to X_s\gamma$



Optical Thm: $\Gamma \sim \mathrm{Im} \int d^4x \, e^{-iq\cdot x} \langle B | T\{J^\dagger_\mu(x) J^\mu(0)\} | B \rangle$



	$P_X^2 = m_B(m_B - 2E_\gamma)$
standard OPE	$\sim m_B^2$
endpoint region	$\sim m_B \Lambda_{QCD}$
resonance region	$\sim \Lambda_{QCD}^2$

For EndPoint: $E_\gamma \gtrsim 2.2\,\mathrm{GeV}$, X_s collinear, B usoft, $\lambda = \sqrt{\frac{\Lambda_{QCD}}{m_B}}$

We want to prove that the
Decay rate is given by factorized form

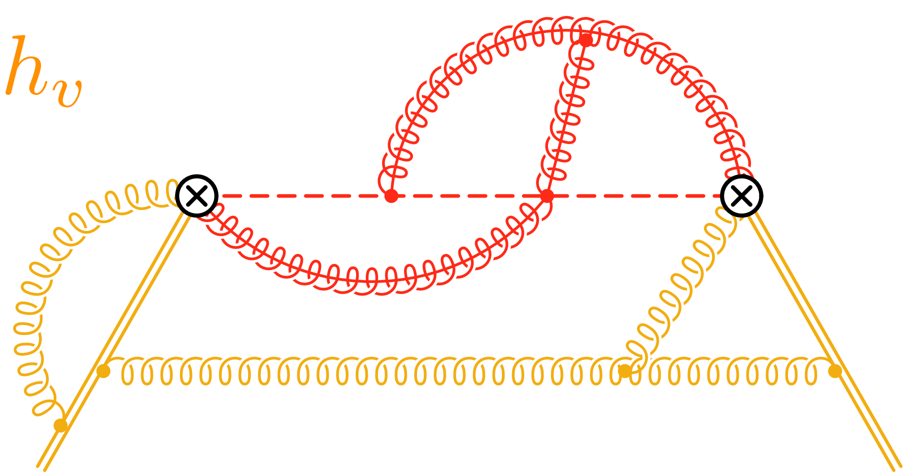
$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = H(m_b,\mu) \int_{2E_\gamma - m_b}^{\bar\Lambda} dk^+ \, S(k^+,\mu) \, J(k^+ + m_b - 2E_\gamma, \mu)$$

Match: $\bar{s}\Gamma_\mu b \to e^{i(m_b v - \mathcal{P})\cdot x} C(\bar{\mathcal{P}}) \bar{\xi}_{n,p} W \gamma_\mu^\perp P_L h_v$

$$T_\mu^\mu = \int d^4x \, e^{i(m_b \frac{\bar{n}}{2} - q)\cdot x} \left\langle B \middle| T J_{\text{eff}}^\dagger(x) J_{\text{eff}}(0) \middle| B \right\rangle$$

label conservation
$\bar{\mathcal{P}} \to m_b$

Factor usoft: $\bar{\xi}_n W \Gamma_\mu h_v \to \bar{\xi}_n W \Gamma_\mu Y_n^\dagger h_v$



$$T_\mu^\mu = \left|C(m_b)\right|^2 \int d^4x e^{i(m_b \frac{\bar{n}}{2} - q)\cdot x} \left\langle B \middle| T[\bar{h}_v Y](x)[Y^\dagger h_v](0) \middle| B \right\rangle$$

$$\times \left\langle 0 \middle| T[W^\dagger \xi_n](x)[\bar{\xi}_n W](0) \middle| 0 \right\rangle \times [\Gamma_\mu \otimes \Gamma^\mu]$$

$$= \left|C(m_b)\right|^2 \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(m_b \frac{\bar{n}}{2} - q - k)\cdot x} \left\langle B \middle| T[\bar{h}_v Y](x)[Y^\dagger h_v](0) \middle| B \right\rangle$$

$$\times J_P(k) \times [\Gamma_\mu \otimes \Gamma^\mu]$$

Convolution $\quad J_P(k) = J_P(k^+)$

$$\mathrm{Im}\,T^\mu_\mu = \left|C(m_b)\right|^2 \int d^4x \int \frac{d^4k}{(2\pi)^4}\, e^{i(m_b \frac{\bar{n}}{2} - q - k)\cdot x} \left\langle B \middle| T[\bar{h}_v Y](x)[Y^\dagger h_v](0) \middle| B \right\rangle$$
$$\times \mathrm{Im} J_P(k^+)$$
$$= \left|C(m_b)\right|^2 \int dk^+ \left[\int \frac{dx^-}{4\pi}\, e^{i(m_b - 2E_\gamma - k^+)x^-/2} \left\langle B \middle| T[\bar{h}_v Y](x)[Y^\dagger h_v](0) \middle| B \right\rangle \right]$$
$$\times \mathrm{Im} J_P(k^+)$$
$$= \left|C(m_b)\right|^2 \int dk^+ S(2E_\gamma - m_b + k^+)\mathrm{Im} J_P(k^+)$$

as desired